$\vec{r} = \langle x, y, z \rangle \quad \vec{\Delta r} = \langle \Delta x, \Delta Y, \Delta z \rangle \quad d\vec{r} = \langle dx, dy, dz \rangle$

C3: ~~Q202~~ Q201 (Chapter 13 – Vector Value Functions): **Lesson 1**

Main Skills (BC REVIEW) – Language New:

Consider the smooth parametrization of curve C: $x = x(t) \quad y = y(t) \quad z = z(t)$

- ☐ Find the *vector equation* of a curve (wire) C in $\Re^3$. $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
- ☐ Find the *vector equation* of a line tangent to a curve C in $\Re^3$. $\boxed{\mathbf{r}_T(t_0) = \mathbf{r}_0 + \mathbf{r}'(t_0)t}$
- ☐ Find the length of a curve C in $\Re^3$. $L_C = \int_C |\mathbf{r}'(t)| dt$

① $\vec{r}(t)$ is the vector function form of curve C

$\rightarrow \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad t \in D$

② $\vec{r}'(t)$ is the vector function that represents slope of tangent to $\vec{r}(t)$ at t

$\rightarrow \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \frac{d\vec{r}}{dt}$ with $d\vec{r} = \langle dx, dy, dz \rangle$

③ $\hat{T}(t)$ is unit vector in direction of $\vec{r}'(t)$

$\rightarrow \hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ with $|\vec{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

④ s = length of curve from $t=a$ to $t=b$ (on Curve C)

$\rightarrow L_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\vec{r}'(t)|\, dt$

⑤ s as a variable function of t:

$\rightarrow s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du = \int_a^t |r'(u)|\, du \quad \therefore \boxed{s = s(t) = \int_a^t |r'(u)|\, du}$

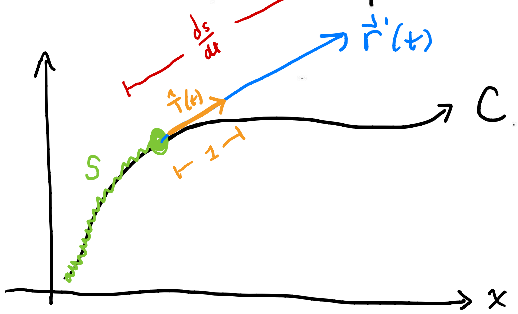
⑥ $\frac{ds}{dt}$ is rate of change in s, w.r.t. t:

$\rightarrow \frac{ds}{dt} = s'(t) = \frac{d}{dt} \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \overset{\text{FTC I}}{=} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = |r'(t)|$

$\therefore \frac{ds}{dt} = |r'(t)| \rightarrow ds = |r'(t)|\, dt = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$|\hat{T}(t)| = 1$

$|\vec{r}'(t)| = \frac{ds}{dt}$



1. Write the vector value function for the curve C: $y = 2x + 1$

$C: x = t \quad y = 2t + 1 \quad t \in \mathbb{R}$

$r(t) = \langle t, 2t+1 \rangle \quad t \in \mathbb{R}$

2. Write the vector value function for the curve C: $x^2 + y^2 = 4$

$C: x = 2\cos t \quad y = 2\sin t \quad 0 \le t \le 2\pi$

$r(t) = \langle 2\cos t, 2\sin t \rangle; \quad 0 \le t \le 2\pi$

3. Write the vector value function for the line curve C that passes through the points $(1,-2,3)$ and $(0,5,-2)$. $\vec{PQ} = \langle -1, 7, -5 \rangle$

$\vec{r}(t) = \langle 1, -2, 3 \rangle + \langle -1, 7, -5 \rangle t$

$\vec{r}(t) = \langle 1 - t, -2 + 7t, 3 - 5t \rangle \quad t \in \mathbb{R}$

4. Write the vector value function for the line segment curve C that runs from the point $(1,-2,3)$ to $(0,5,-2)$.

segment !

$\vec{r}(t) = \langle 1 - t, -2 + 7t, 3 - 5t \rangle \quad 0 \le t \le 1$

5. Write the vector value function for the curve C of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

$C: x = \cos t \quad y = \sin t \quad z = 2 - \sin t$

$\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle \quad 0 \le t \le 2\pi$

6. Write the vector value function for the curve C of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

$C: x = \cos t \quad y = 2\sin t \quad z = 2 - \cos t - 2\sin t$

$\therefore \vec{r}(t) = \langle \cos t, 2\sin t, 2 - \cos t - 2\sin t \rangle \quad 0 \le t \le 2\pi$

7. Suppose a curve C is defined as: $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t \rangle; \ t \ge 0$

A. What is the shape of C? helix

B. Write an equation of the line tangent to C at $t = \frac{\pi}{2}$:

$r'(t) = \langle -3\sin t, 3\cos t, 1 \rangle$

$r'(\frac{\pi}{2}) = \langle -3, 0, 1 \rangle$

$r(\frac{\pi}{2}) = \langle 0, 3, \pi/2 \rangle$

$\therefore \vec{r}_T(t) = \langle -3t, 3, \frac{\pi}{2} + t \rangle \quad t \ge 0$

C. Find the length of C from $t = \frac{\pi}{2}$ to $t = 2\pi$.

should be 9

$|r'(t)| = \sqrt{9\sin^2 t + 3\cos^2 t + 1} = \sqrt{10}$

$L_{\frac{\pi}{2}}^{2\pi} = \int_{\pi/2}^{2\pi} |r'(t)|\,dt = \int_{\pi/2}^{2\pi} \sqrt{10}\,dt = \sqrt{10}\,[t]_{\pi/2}^{2\pi} = \frac{3\sqrt{10}}{2}\pi$

8. Suppose a curve C is defined as: $\mathbf{r}(t) = \left\langle \ln t, 2\sqrt{t}, t^2 \right\rangle$ $t > 0$

$r'(t) = \langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \rangle$

A. Write an equation of the line tangent to C at $t = 1$:

$r'(1) = \langle 1, 1, 2 \rangle$

$r(1) = \langle 0, 2, 1 \rangle$

$\therefore \vec{r}_T(t) = \langle t, t+2, 2t+1 \rangle \quad t \in \mathbb{R}$

note $\mathbb{R}$: it's b/c a line will always extend in all direction

B. Find the length of C from $t = 1$ to $t = 5$.

$$L_1^5 = \int_1^5 |r'(t)|\, dt = \int_1^5 \sqrt{\frac{1}{t^2} + \frac{1}{t} + 4t^2}\, dt$$

$$= \int_1^5 \frac{\sqrt{1 + t + 4t^4}}{t}\, dt \approx 24.309$$

9. Suppose a curve C is defined as the intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

$C: x = \cos t \quad y = 2\sin t \quad z = 2 - \cos t - 2\sin t$

$r(t) = \langle \cos t, 2\sin t, 2 - \cos t - 2\sin t \rangle$

$r'(t) = \langle -\sin t, 2\cos t, \sin t - 2\cos t \rangle$

$r'(1) = \langle -\sin 1, 2\cos 1, \sin 1 - 2\cos 1 \rangle$

$r(1) = \langle \cos 1, 2\sin 1, 2 - \cos 1 - 2\sin 1 \rangle$

A. Write an equation of the line tangent to C at $t = 1$:

$\vec{r}_T(1) = \langle \cos 1 - t\sin 1, 2\sin 1 + 2t\cos 1, 2 - \cos 1 - 2\sin 1 + t(\sin 1 - 2\cos 1) \rangle \quad t \in \mathbb{R}$

B. Find the length of C.

$$L_0^{2\pi} = \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 t + (\sin t - 2\cos t)^2}\, dt = 13.519$$

C3: Q202 (Chapter 13 – Vector Value Functions): **Lesson 2**

1. Theorem: If $\mathbf{r}(t)$ is differentiable and $|\mathbf{r}(t)|$ is constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for every t in the domain of $\mathbf{r}(t)$.

Proof: Given $|\vec{r}(t)| = c$

$$\vec{r}\cdot\vec{r} = |\vec{r}|^2 = c^2$$

$$(\vec{r}\cdot\vec{r})' = (c^2)'$$

same prod. rule applies

$$\vec{r}\cdot\vec{r}' + \vec{r}'\cdot\vec{r} = 0$$

$$2(\vec{r}\cdot\vec{r}') = 0$$

$$\vec{r}\cdot\vec{r}' = 0$$

$$\therefore\ \vec{r} \perp \vec{r}'$$

2. Language Review:

$$\vec{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$s(t) = \int_a^t |\vec{r}'(u)|\,du \longrightarrow \frac{ds}{dt} = |\vec{r}'(t)|$$

$$\hookrightarrow \text{ with } ds = |\vec{r}'(t)|\,dt = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$\hat{T}(t) = \frac{r'(t)}{|r'(t)|} \qquad \boxed{\hat{N} = \frac{T'(t)}{|T'(t)|}} = \underline{\text{unit vector normal to } C}$$

(proven below)

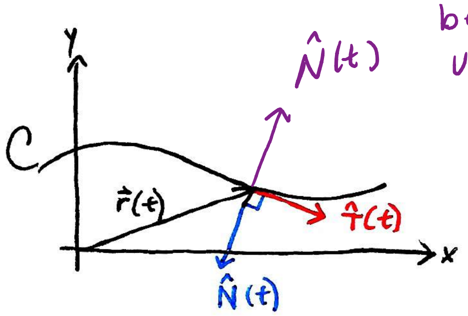
3. Prove that $\mathbf{T} \perp \mathbf{N}$ $\left(\text{OR } T \perp \frac{T'}{|T'|}\right)$

Given $|T| = 1$ (constant)

$$\therefore\ T \perp T'$$

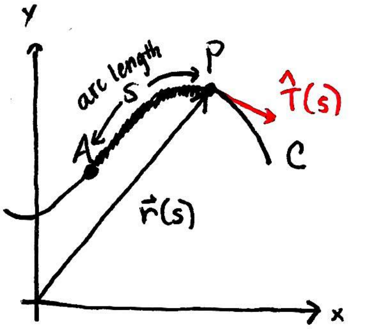
$$\therefore\ T \perp \frac{T'}{|T'|} \quad \text{scalar multiple}$$

$$\therefore\ T \perp N$$



normal vector should be pointing up, not down

4. Parametrization with respect to arc-length s (concept and language):



$x = x(s) \quad y = y(s) \quad z = z(s)$

$$\vec{r}(s) = \langle x(t(s)),\ y(t(s)),\ z(t(s)) \rangle$$

$$\vec{r}(s) = \langle x(s),\ y(s),\ z(s) \rangle$$

$$\vec{r}\,'(s) = \left\langle \frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds} \right\rangle$$

$$|\vec{r}\,'(s)| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

$$= \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{\sqrt{(ds)^2}} = \frac{ds}{ds} = 1$$

$$\hat{T}(s) = \frac{\vec{r}\,'(s)}{|\vec{r}\,'(s)|} = \frac{\vec{r}\,'(s)}{1} = \vec{r}\,'(s)$$

5. Reparametrization $\mathbf{r}(t) \to \mathbf{r}(s)$:

A. reparametrize $\mathbf{r}(t) = \langle 4t-3, 3t-5 \rangle$ D: $t \geq 0$ in terms of arc-length.

1. Solve for t in terms of s:

$x = 4t - 3 \quad y = 3t - 5$

$x' = 4 \quad y' = 3$

$$s(t) = \int_0^t \sqrt{4^2 + 3^2}\, du = [5u]_0^t = 5t$$

$\therefore\ s = 5t \rightarrow t = \frac{1}{5}s$

remember $|\vec{r}\,'(s)| = 1$

2. Substitute: $\vec{r}(s) = \langle \frac{4}{5}s - 3,\ \frac{3}{5}s - 5 \rangle \quad s \geq 0$

Ex:

	x	y
$t = 0$	-3	-5
$t = 3/4$	0	$-11/4$
$t = 1$	1	-2

	x	y
$s = 0$	-3	-5
$s = 15/4$	0	$-11/4$
$s = 5$	1	-2

So: $\frac{dr}{dt} = \langle 4, 3 \rangle$

$\frac{dr}{ds}\ \cancel{\frac{ds}{dt}} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

same direction
diff. magnitude

B. Reparametrize $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ D: $t \geq 0$ in terms of arc-length.

$x = \cos t \qquad y = \sin t \qquad z = t$

$x' = -\sin t \qquad y' = \cos t \qquad z' = 1$

$$s(t) = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1}\, du = \sqrt{2} \int_0^t du = \sqrt{2}\,[u]_0^t = t\sqrt{2}$$

$$s = t\sqrt{2} \rightarrow t = \frac{s}{\sqrt{2}}$$

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle \quad s \geq 0$$

C. Reparametrize $\mathbf{r}(t) = \langle e^{2t}\cos(2t), 2, e^{2t}\sin(2t) \rangle$ D: $t \geq 0$ in terms of arc-length.

$x = e^{2t}\cos 2t \qquad x' = e^{2t}(-2\sin 2t) + (2e^{2t})(\cos 2t)$

$y = 2 \qquad y' = 0$

$z = e^{2t}\sin 2t \qquad z' = e^{2t}(2\cos 2t) + (2e^{2t})(\sin 2t)$

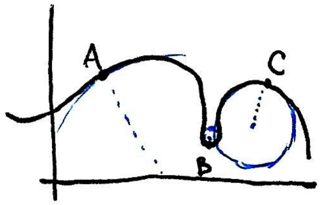
$$s(t) = \int_0^t \sqrt{(-2e^{2u}\sin 2u + 2e^{2u}\cos 2u)^2 + 0^2 + (2e^{2u}\cos 2u + 2e^{2u}\sin 2u)^2}\, du$$

$$s(t) = 2\sqrt{2} \int_0^t e^{2u}\, du = \sqrt{2}(e^{2t} - 1)$$

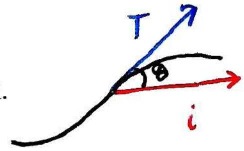
$$s = \sqrt{2}(e^{2t} - 1) \qquad t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\vec{r}(s) = \left\langle \left(\frac{s}{\sqrt{2}} + 1\right)\cos\left[\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right], 2, \left(\frac{s}{\sqrt{2}} + 1\right)\sin\left[\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right] \right\rangle$$

6. Curvature in R^2:



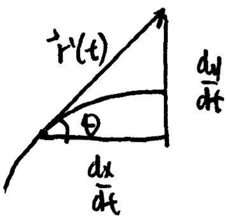
Concept: $k = \left|\dfrac{d\theta}{ds}\right|$ where θ is the angle between **T** and **i** and s is the arc length.



Formula when $x = x(t)$, $y = y(x)$: $k(t) = \left|\dfrac{x'(t)y''(t) - y'(t)x''(t)}{\left[(x'(t))^2 + (y'(t))^2\right]^{\frac{3}{2}}}\right| = \left|\dfrac{d\theta}{ds}\right|_t$

Formula when $y = f(x)$: $k(x) = \left|\dfrac{f''(x)}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}}\right| = \left|\dfrac{d\theta}{ds}\right|_x$

$\dot{x} = \dfrac{dx}{dt}$ $\quad \dot{y} = \dfrac{dy}{dt}$



$$\frac{d\theta}{ds} = \frac{\frac{d\theta}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\theta}{dt}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

now find $d\theta/dt$

← Newton's notation

$$\tan\theta = \frac{\dot{y}}{\dot{x}} \xrightarrow{\text{implicit}} \sec^2\theta \, \frac{d\theta}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}$$

(Pyth. identity)

$$\frac{d\theta}{dt} = \frac{1}{1 + \tan^2\theta} \cdot \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{1}{1 + (\dot{y}/\dot{x})^2} \cdot \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}$$

$$\frac{d\theta}{ds} = \frac{\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^1}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

(all in terms of t)

(Substitute)

7. Curvature in R^3:

Concept: $k = \left|\frac{d\mathbf{T}}{ds}\right|$

Formulas: $k(s) = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{\frac{d\hat{T}}{ds}}{\frac{ds}{ds}}\right| = \left|\frac{T'(s)}{|r'(s)|}\right| = \frac{|r'(s)|}{1} = |T'(s)|$

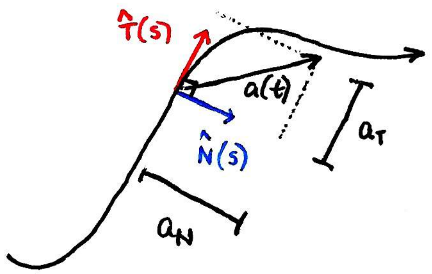
$\therefore \boxed{k(s) = |T'(s)|}$

$k(t) = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|T'(t)|}{||r'(t)||} = \frac{|T'(t)|}{|r'(t)|}$

OR alt: $k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ (see next lesson)

C3: Q202 (Chapter 13 – Vector Value Functions): **Lesson 3**

1. Lesson Overview and Diagram



$\vec{r} = \langle x, y, z \rangle$

$\vec{r}\,' = \vec{v} = \langle x', y', z' \rangle$

$\vec{r}\,'' = \vec{a} = \langle x'', y'', z'' \rangle$

$$\rightarrow \vec{r}\,''(t) = \vec{a}(t) = \underbrace{a_T \hat{T}(s)}_{\text{tangential component}} + \underbrace{a_N \hat{N}(s)}_{\text{normal component}}$$

scalar · unit vector

a_T and a_N as scalars:

$$a_T = \frac{d^2 s}{dt^2} \qquad a_N = k\left(\frac{ds}{dt}\right)^2$$

We will show:

Tangential component of acc $a_T = \dfrac{d^2 s}{dt^2}$

Normal component of acc $a_N = k\left(\dfrac{ds}{dt}\right)^2$

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|} \qquad a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$$

2. SET OF VECTOR PROPERTIES

VECTOR PROPERTIES FOR CH 13 LESSON 3

① $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ [CH 12.3]

② $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ [CH 12.3]

③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ [CH 12.3]

④ $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = a \cdot (k\vec{b})$ [CH 12.3]
$k = \text{scalar}$

⑤ $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ [CH 12.3]

⑥ $\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$ [CH 12.3]

⑦ $\text{Comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ [CH 12.3]

⑧ $\text{Proj}_{\vec{a}} \vec{b} = \left(\text{Comp}_{\vec{a}} \vec{b}\right) \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$ [CH 12.3]

⑨ $\vec{a} \times \vec{b}$ $\perp$ to both $\vec{a}$ and $\vec{b}$ [CH 12.4]

⑩ $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ [CH 12.4]

⑪ $\vec{a} \parallel \vec{b}$ iff $\vec{a} \times \vec{b} = \vec{0}$ [CH 12.4]

⑫ A: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ [CH 12.4]

B: $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \rightarrow (c_1\vec{a}) \times (c_2\vec{b}) = c_1c_2(\vec{a} \times \vec{b})$

C: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

D: $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

E: $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

F: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$\boxed{13.2}$

$\frac{d}{dt}[\vec{u} + \vec{v}] = u' + v'$ $\quad$ $\frac{d}{dt}[\vec{u} \cdot \vec{v}] = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$ Product

$\frac{d}{dt}[c\vec{u}] = cu'$ $\quad$ $\frac{d}{dt}[\vec{u} \times \vec{v}] = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$ Product

$\frac{d}{dt}[f(t)\vec{u}] = f'(t)\vec{u} + f(t)\vec{u}'$ $\quad$ $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$ chain

3. Prove Property - 13B (in R^2)

$$\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] = \frac{d}{dt}\left[\langle x_1(t), y_1(t)\rangle \cdot \langle x_2(t), y_2(t)\rangle\right]$$ Write components

$$= \frac{d}{dt}\left[x_1(t)x_2(t) + y_1(t)y_2(t)\right]$$ Dot product

$$= \underline{x_1(t)x_2'(t)} + \underline{x_1'(t)x_2(t)} + \underline{y_1(t)y_2'(t)} + \underline{y_1'(t)y_2(t)}$$ Derivative (use PR to treat as functions)

$$= \langle x_1, y_1\rangle \cdot \langle x_2', y_2'\rangle + \langle x_1', y_1'\rangle \cdot \langle x_2, y_2\rangle = uv' + u'v$$ Group

4. Prove $\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}(t)\times\mathbf{r}''(t)$

$$\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \underbrace{r'(t)\times r'(t)}_{\text{parallel b/c same vector}} + r(t)\times r''(t)$$ Product rule for vectors

$$= 0 + r(t)\times r''(t) = r(t)\times r''(t) \quad \text{QED.}$$

5. Prove $\frac{d}{dt}|\mathbf{r}(t)| = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}$

manipulate Defn of $r\cdot r$ Chain rule + product rule

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{d}{dt}\left[|r(t)|^2\right]^{1/2} = \frac{d}{dt}[r\cdot r]^{1/2} = \frac{1}{2}(r\cdot r)^{-1/2}(r\cdot r' + r\cdot r')$$

$$= \frac{2(r\cdot r')}{2|r|^{2\cdot 1/2}} = \frac{r(t)\cdot r'(t)}{|r'(t)|}$$

simplify

6. Prove $k\mathbf{a}\cdot m\mathbf{b} = km(\mathbf{a}\cdot\mathbf{b})$

$$k\mathbf{a}\cdot m\mathbf{b} = k(\vec{a}\cdot m\vec{b}) = km(\vec{a}\cdot\vec{b})$$

7. BERMEL TRUE / FALSE UNDERSTANDING

$\vec{r}(t) = \vec{r}(s) \rightarrow$ * same physical curve

$\vec{r}'(t) \neq \vec{r}'(s)$

$$\frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{r}'(s)}{|\vec{r}'(s)|} \rightarrow \hat{T}(t) = \hat{T}(s)$$

$$\hat{T}'(t) \neq \hat{T}'(s)$$

$$|\hat{T}'(t)| \neq |\hat{T}'(s)| = K$$

8. Prove $\mathbf{T}'(s) = K\mathbf{N}(s)$

$$K = |T'(s)| \quad \text{and} \quad N(s) = \frac{T'(s)}{|T'(s)|}$$

$$N(s) = \frac{T'(s)}{K} \rightarrow T'(s) = K\cdot N(s)$$

~~sec~~ $\frac{ds}{dt} = |r'(t)|$ which is a scalar function

9. Prove $v(t) = T(s)\frac{ds}{dt}$

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \rightarrow \vec{r}'(t) = \vec{T}(t)\,|\vec{r}'(t)|$ Multiply

$\vec{v}(t) = \vec{T}(t)\,|\vec{r}'(t)|$ $\vec{r}'(t) = v(t)$

$\vec{v}(t) = \vec{T}(s)\frac{ds}{dt}$ QED. $\vec{T}(s) = \vec{T}(t)$

10. Prove $a(t) = \underbrace{\frac{d^2s}{dt^2}}_{=a_T} T(s) + \underbrace{K\left(\frac{ds}{dt}\right)^2}_{=a_N} N(s)$

that is, show:

USE : $\frac{d}{dt}T(s)$

$\vec{a}(t) = [v(t)]' = \frac{d}{dt}\left[\vec{T}(s)\frac{ds}{dt}\right]$ Substitute $v(t) = \hat{T}(s)\frac{ds}{dt}$

$= \hat{T}(s)\cdot\frac{d}{dt}\left(\frac{ds}{dt}\right) + \boxed{\hat{T}'(s)}\frac{ds}{dt}$ Product rule

$= \frac{d^2s}{dt^2}\hat{T}(s) + \frac{ds}{dt}\boxed{\left(\frac{d\hat{T}}{ds}\cdot\frac{ds}{dt}\right)}$ Chain rule on $\hat{T}'(s) = \frac{d}{dt}T(s)$

$= \frac{d^2s}{dt^2}\hat{T}(s) + \left(\frac{ds}{dt}\right)^2\hat{T}'(s)$ Group $\frac{ds}{dt}$ and $\hat{T}'(s)$ notation

$= \frac{d^2s}{dt^2}\hat{T}(s) + \left(\frac{ds}{dt}\right)^2 k\hat{N}(s)$ Substitute $\hat{T}'(s) = k\hat{N}(s)$

$$\therefore\ \vec{a}(t) = \underbrace{\frac{d^2s}{dt^2}}_{=a_T}\hat{T}(s) + \underbrace{k\left(\frac{ds}{dt}\right)^2}_{=a_N}\hat{N}(s)$$

11. Prove $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$

$$r'(t)\cdot r''(t) = \underbrace{\left[\frac{ds}{dt}\hat{T}(s)\right]}_{\vec{v}(t)} \cdot \underbrace{\left[\frac{d^2s}{dt^2}\hat{T}(s) + k\left(\frac{ds}{dt}\right)^2\hat{N}(s)\right]}_{\vec{a}(t)}$$

$$= \left(\frac{ds}{dt}\hat{T}(s)\right)\cdot\left(\frac{d^2s}{dt^2}\hat{T}(s)\right) + \left(\frac{ds}{dt}\hat{T}(s)\right)\cdot\left(k\left(\frac{ds}{dt}\right)^2\hat{N}(s)\right)$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2}\underbrace{\left(\hat{T}(s)\cdot\hat{T}(s)\right)}_{=1 \text{ b/c } = |\hat{T}(s)| = 1^2} + \cancel{k\frac{ds}{dt}\left(\frac{ds}{dt}\right)^2\underbrace{\left(\hat{T}(s)\cdot\hat{N}(s)\right)}_{=0 \text{ b/c orthogonal}}} \to 0$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2} = |r'(t)|\cdot a_T$$

$$\therefore \quad a_T = \frac{r'(t)\cdot r''(t)}{|r'(t)|} \quad \text{QED}$$

12. Prove $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$

$$\left\{\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}\right\}$$

$$\left\{k_1\vec{a}\times k_2\vec{b} = k_1k_2(\vec{a}\times\vec{b})\right\}$$

$$r'(t)\times r''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\underbrace{\left[T(s)\times T(s)\right]}_{\langle 0,0,0\rangle \text{ parallel}} + K\left(\frac{ds}{dt}\right)^3\underbrace{\left[T(s)\times N(s)\right]}_{\text{vector}}$$

$$= K\left(\frac{ds}{dt}\right)^3\left[T(s)\times N(s)\right]$$

$$|r'(t)\times r''(t)| = \left|K\left(\frac{ds}{dt}\right)^3(T(s)\times N(s))\right| = K\left(\frac{ds}{dt}\right)^3\underbrace{|T(s)\times N(s)|}_{\text{area of unit square (parallelogram) equals 1}}$$

$$= K\left(\frac{ds}{dt}\right)^3 = K\left(\frac{ds}{dt}\right)^2\frac{ds}{dt}$$

$$= a_N\left(\frac{ds}{dt}\right) = a_n|\vec{r}(t)|$$

$$\therefore \quad a_n = \frac{|r'(t)\times r''(t)|}{|r'(t)|} \quad \text{QED}$$